

# On Decay of Correlations for Exclusion Processes with Asymmetric Boundary Conditions

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## Abstract

We consider a symmetric exclusion process on a discrete interval of  $S$  points with various boundary conditions at the endpoints. We study the asymptotic decay of correlations as  $S \rightarrow \infty$ . The main result is asymptotic independence of a stationary distribution when the points are far away from each other. We develop a new recurrent probabilistic approach which is an alternative to Derrida's algebraic technique.

## 1 Introduction

Exclusion processes are a long-standing popular subject in both mathematics and theoretical physics. They are a particular case of processes with local interaction on a lattice. The interest to them is mainly due to the fact that exclusion processes are the simplest nontrivial model for collisions in a multiparticle system. Using them, heat conduction [1], viscosity [2, 3], quantum ferromagnet [4], nonequilibrium process [5], etc., models are constructed. Even irrespective of physics, these processes are natural probabilistic objects. The first systematic exposition of the relevant theory was given in the well-known monograph [6].

The simplest exclusion process is a symmetric exclusion process on the integer lattice  $\mathbb{Z}$ . This process is well known to have a continuum of (spatially) uniform invariant measures, which are mixtures of Bernoulli measures. The same holds for an exclusion process on a finite segment of the lattice if boundary conditions are empty. For other boundary conditions, as a rule, it is easy to show that there are no invariant Bernoulli measures. Using a matrix method (ansatz; see [5, 7]), one can obtain an explicit form of correlation functions for such a process. This would imply that the invariant measure is asymptotically Bernoullian. It should be noted that this powerful algebraic method, similar to the famous Bethe ansatz, is rather cumbersome, is not always mathematically well-grounded, and has substantial restrictions on its range of applicability. Namely, we are unaware of any application where jumps of particles can be longer than 1. It is also important that the probabilistic nature of the method is absolutely unclear.

In the present paper, we propose another—simple and natural from the probability theory viewpoint—approach, which can be extended to jumps of lengths greater than 1, as well as to other boundary conditions. Being quite different from the Bethe–Derrida methods, our approach has something in common with them, namely, a certain recursive procedure. Here we demonstrate the idea of our approach in the simplest situation. Various generalizations will be considered in further publications.

## 2 Problem Statement and the Result

We consider a simple symmetric exclusion process on an interval  $I_S = \{0, 1, \dots, S, S+1\}$  of a 1D lattice. The state space for this continuous-time finite Markov chain

$$(\xi_t(0), \xi_t(1), \dots, \xi_t(S+1)), \quad \xi_t(i) = 0, 1,$$

is the set  $\{0, 1\}^{I_S}$ . Jumps are defined as follows. For any time interval  $[t, t+dt]$  and any pair  $s, s+1$ , where  $s = 0, 1, \dots, S$ , the nodes  $s, s+1$  interchange their values with probability  $\lambda dt$  (independently for different  $s$ ), so that

$$\xi_{t+dt}(s) = \xi_t(s+1), \quad \xi_{t+dt}(s+1) = \xi_t(s).$$

This defines a generator of a Markov process that we call a process with empty boundary conditions. However, we shall consider boundary conditions:

$$\xi_t(0) \equiv 0, \quad \xi_t(S+1) \equiv 1.$$

More precisely, this means that  $\xi_0(0) = 0$  and interchange between the points  $0, 1 \in I_S$  reduces to the following: the pair  $\xi_t(0) = 0, \xi_t(1) = 1$  becomes  $\xi_{t+dt}(0) = 0, \xi_{t+dt}(1) = 0$  with probability  $\lambda dt$ . The same for the points  $S, S+1 \in I_S$ :  $\xi_0(S+1) = 1$  and the pair  $\xi_t(S) = 0, \xi_t(S+1) = 1$  becomes  $\xi_{t+dt}(S) = 1, \xi_{t+dt}(S+1) = 1$  with probability  $\lambda dt$ .

Denote by  $\pi(n_0, n_1, \dots, n_{S+1}) = \pi^{(S)}(n_0, n_1, \dots, n_{S+1})$  the stationary measure of this Markov chain. We are interested in the functions

$$m_k^{(S)}(x_1, x_2, \dots, x_k) = \pi^{(S)}(n_{x_1} = 1, n_{x_2} = 1, \dots, n_{x_k} = 1), \quad x_1 < \dots < x_k.$$

For the empty boundary conditions, the process is reducible (because of the conservation of the number of particles, i.e., the number of ones), and any stationary measure is a mixture of Bernoulli measures, i.e., measures that are products of  $S+2$  equal measures. As is easy to show, in the case of our boundary conditions the process is irreducible (the number of particles is not preserved now), and its invariant measure is not a product of independent measures. Our goal is to prove the following correlation decay property for this process.

**Theorem 1** *For all  $0 \leq x \leq S+1$ , we have*

$$m_1^{(S)}(x) = \frac{x}{S+1}.$$

For any  $0 < x_1 = x_1(S) < x_2 = x_2(S) < S+1$  such that  $\frac{x_k(s)}{S+1} \rightarrow \alpha_k, k = 1, 2$  as  $S \rightarrow \infty$ , we have

$$m_2^{(S)}(x_1, x_2) \rightarrow_{S \rightarrow \infty} \alpha_1 \alpha_2$$

### 3 Proof

The proof consists of two parts. The first part, reduction to a dual process (which is easier to study), is rather standard but cumbersome. The second part, analysis of the dual process, is original.

#### 3.1 Moment Closeness

First we obtain equations for moments of the process  $\xi_t$ :

$$m_k(t; \{x_1, \dots, x_k\}) = \mathbf{E} \prod_{i=1}^k \xi_t(x_i), \quad x_1, \dots, x_k \in I_S, \quad k = 1, 2, \dots$$

For a process with values 0 and 1, these moments uniquely determine marginal distributions

$$Pr \left( \bigcap_{x \in S^o} \{\xi_t(x) = \theta_x\} \right)$$

and, of course, vice versa. These equations possess the property of moment or marginal closeness [8], which makes it possible to construct a convenient dual process. In what follows, we assume for simplicity that  $\lambda = 1$ .

We introduce the notion of a cluster. Let a set  $Q \subset \mathbb{N}$  be given. A subset  $(k_1 < \dots < k_m) = K \subseteq Q$  is called a cluster if  $k_{i+1} - k_i = 1$  for all  $i = 1, \dots, m-1$  and, moreover,  $\{k_1 - 1\} \notin Q$  and  $\{k_m + 1\} \notin Q$ . In other words, a cluster is a maximal chain of successive elements of  $Q$ .

**Lemma 2** Consider a set  $X = (x_1, \dots, x_k)$  such that  $0 < x_1 < \dots < x_k < S+1$ . Let  $X = \bigcup_{j=1}^p K_j$ , i.e.  $X$  is a union of clusters  $K_j = (x_{i_1}^{(j)}, \dots, x_{i_{q(j)}}^{(j)})$ . Then for any  $X$  we have the equation

$$\begin{aligned} \frac{d}{dt} m_k(t, \{x_1, \dots, x_k\}) &= \sum_{j=1}^p m_k(t, \{x_1, \dots, x_{i_1}^{(j)} - 1, \dots, x_k\}) \\ &+ \sum_{j=1}^p m_k(t, \{x_1, \dots, x_{i_{q(j)}}^{(j)} + 1, \dots, x_k\}) - 2p m_k(t, \{x_1, \dots, x_k\}) \end{aligned}$$

with boundary conditions

$$\begin{aligned} m_k(t, \{0, x_2, \dots, x_k\}) &= 0, \\ m_k(t, \{x_1, \dots, x_{k-1}, S+1\}) &= m_{k-1}(t, \{x_1, \dots, x_{k-1}\}). \end{aligned}$$

Proof. First we derive the equation for a one-point function. The moment closeness property is based on the linearity of the conditional probability

$$Pr(\xi_{t+dt}(x) = 1 \mid \xi(x-1) = \alpha, \xi_t(x) = \beta, \xi_t(x+1) = \gamma) = \alpha dt + \beta(1-2dt) + \gamma dt.$$

By the total probability formula,

$$\begin{aligned} Pr(\xi_{t+dt}(x) = 1) &= \sum_{\alpha, \beta, \gamma=0}^1 (dt(\alpha + \gamma - 2\beta) + \beta) Pr(\xi_t(x-1) = \alpha, \xi_t(x) = \beta, \xi_t(x+1) = \gamma) \\ &= dt \left( Pr(\xi_t(x-1) = 1) + Pr(\xi_t(x+1) = 1) - 2Pr(\xi_t(x) = 1) \right) + Pr(\xi_t(x) = 1), \end{aligned}$$

whence we get

$$\frac{d}{dt} m_1(t; \{x\}) = m_1(t; \{x+1\}) + m_1(t; \{x-1\}) - 2m_1(t; \{x\}).$$

Now we pass to correlation functions of an arbitrary order  $k$ . Let  $X$  consist of clusters of length 1 only. Since a change during time  $dt$  can only occur at one of the points  $x_i$ , similar computations for each of them yield

$$\begin{aligned} \frac{d}{dt} m_k(t; \{x_1, \dots, x_k\}) &= \sum_{i=1}^k \left( m_k(t; \{x_1, \dots, x_i+1, \dots, x_k\}) \right. \\ &\quad \left. + m_k(t; \{x_1, \dots, x_i-1, \dots, x_k\}) \right) - 2km_k(t; \{x_1, \dots, x_k\}). \end{aligned}$$

Now consider the situation with clusters of larger lengths. As above, it suffices to consider the case of only one such cluster. Thus, let  $K_j = (x_{i_1}^{(j)}, \dots, x_{i_q(j)}^{(j)})$  be the  $j$ th cluster in the partition of  $X$ . Changes can only occur at the endpoints of the cluster. Let us write the general form of the conditional probability in this case, omitting the index  $j$ :

$$\begin{aligned} Pr(\xi_{t+dt}(x_{i_1}) = 1, \dots, \xi_{t+dt}(x_{i_q}) = 1 \mid \xi_t(x_{i_1}-1) = \alpha, \\ \xi_t(x_{i_1}) = \beta, \xi_t(x_{i_q}) = \gamma, \xi_t(x_{i_q}+1) = \delta) = \alpha\gamma dt + \beta\gamma(1-2dt) + \beta\delta dt. \end{aligned}$$

Of course, the conditional probability is taken provided that  $\xi_t(x_{i_j}) = 1, \forall j = 2, \dots, q-1$ . Then we obtain

$$\begin{aligned} &Pr(\xi_{t+dt}(x_{i_1}) = 1, \dots, \xi_{t+dt}(x_{i_q}) = 1) \\ &= \sum_{\alpha, \beta, \gamma, \delta=0}^1 (\alpha\gamma dt + \beta\gamma(1-2dt) + \beta\delta dt) Pr(\xi_t(x_{i_1}-1) = \alpha, \xi_t(x_{i_1}) = \beta, \\ &\quad \xi_t(x_{i_1}+1) = 1, \dots, \xi_t(x_{i_q}-1) = 1, \xi_t(x_{i_q}) = \gamma, \xi_t(x_{i_q}+1) = \delta) \\ &= dt \left( m_q(t; \{x_{i_1}-1, \dots, x_{i_q}\}) + m_q(t; \{x_{i_1}, \dots, x_{i_q}+1\}) \right. \\ &\quad \left. - 2m_q(t; \{x_{i_1}, \dots, x_{i_q}\}) \right) + m_q(t; \{x_{i_1}, \dots, x_{i_q}\}), \end{aligned}$$

which implies that for an individual cluster we have the equation

$$\begin{aligned} \frac{d}{dt} m_q(t; \{x_{i_1}, \dots, x_{i_q}\}) &= m_q(t; \{x_{i_1} - 1, \dots, x_{i_q}\}) + m_q(t; \{x_{i_1}, \dots, x_{i_q} + 1\}) \\ &\quad - 2m_q(t; \{x_{i_1}, \dots, x_{i_q}\}). \end{aligned}$$

Taking the sum over clusters, we obtain the lemma. For convenience, we write the right-hand side of the equation as follows:

$$\frac{d}{dt} m_k(t; \{x_1, \dots, x_k\}) = \sum_{j=1}^k \Delta_{x_j}^2 m_k(t; \{x_1, \dots, x_k\}).$$

### 3.2 Dual Process

Denote by  $2^{I_S}$  the set of (finite) subsets of  $I_S$ , including the empty set  $\emptyset$ . Let  $(\xi_t, t \geq 0)$  be the exclusion process defined above. A process  $(A_t, t \geq 0)$  with values in  $2^{I_S}$  is said to be *dual* to the process  $(\xi_t, t \geq 0)$  if for all  $t \geq 0$  we have

$$E \prod_{x \in A_0} \xi_t(x) = E \prod_{x \in A_t} \xi_0(x).$$

In [6, Section VIII.1, Theorem 1.1] it is proved that a *symmetric* exclusion process is *self-dual*. Thus, for instance, analysis of two-point correlation functions can be reduced to consideration of two particles performing an exclusion walk. To extend this result to our boundary conditions, consider the above equations. They lead to the following statement.

**Lemma 3** *A process dual to  $(\xi_t, t \geq 0)$  is an exclusion process  $(A_t, t \geq 0)$  for a finite number of particles with the following modification:*

*[(a)]*

1. *If one of the particles of  $A_t$  touches the boundary 0, the whole configuration dies (we also say that all particles reach the absorbing state 0):*

$$\sigma := \sup\{s > 0 : A_s \cap \{0\} \neq \emptyset\} \implies A_\sigma = \emptyset;$$

2. *Any particle that reaches the boundary  $S + 1$  sticks to it. After that, the other particles walk independently of the stuck particle, whose coordinate remains to be  $S + 1$ .*

Note that after one of the particles reaches the right-hand boundary, other particles either hit 0 or reach the point  $S + 1$ . Therefore, with probability 1, the Markov process thus defined comes in the course of time to one of the absorbing states:  $\emptyset$  or  $\{S + 1, \dots, S + 1\}$ .

As a consequence, we get the following statement.

**Proposition 4** *We have*

$$m_k^{(S)}(A) = \sum_{\ell \in L(A)} Pr(\ell), \quad |A| = k,$$

where  $L(A)$  is the set of all trajectories of the dual process that start in  $A$  and do not hit the boundary 0.

Proof. Let the process  $\eta_t(x) = (\eta_t^1(x_1), \dots, \eta_t^k(x_k)) \in \mathbb{Z}_+^k$  be an exclusive random walk of  $k$  particles in the segment  $[0, S+1]$ , where  $x = (x_1, \dots, x_k) \in \mathbb{Z}_+^k$  is the initial disposition of particles. To prove the duality, it suffices to show that the function  $E \prod_{i=1}^k \xi_0(\eta_t^i(x_i))$  satisfies the same differential equations as the correlation function  $E \prod_{i=1}^k \xi_t(x_i)$ . By the total probability formula, we have

$$E \prod_{i=1}^k \xi_0(\eta_t^i(x_i)) = \sum_{i_1 < \dots < i_k} Pr(\xi_0(i_1) = 1, \dots, \xi_0(i_k) = 1) Pr(\eta_t^1(x_1) = i_1, \dots, \eta_t^k(x_k) = i_k).$$

Denote

$$P_t(i_1, \dots, i_k) := Pr(\eta_t^1(x_1) = i_1, \dots, \eta_t^k(x_k) = i_k).$$

This function satisfies the same equation as the moments, i.e.,

$$\frac{d}{dt} P_t(i_1, \dots, i_k) = \sum_{j=1}^k \Delta_{i_j}^2 P_t(i_1, \dots, i_k).$$

Indeed, for a single particle we have

$$\begin{aligned} P_{t+dt}(i) &= \sum_{j=i-1}^{i+1} Pr(\eta_{t+dt}(x) = i \mid \eta_t(x) = j) Pr(\eta_t(x) = j) \\ &= dt \left( Pr(\eta_t(x) = i-1) + Pr(\eta_t(x) = i+1) \right) + (1-2dt) \P(\eta_t(x) = i). \end{aligned}$$

Hence,

$$\frac{d}{dt} P_t(i) = P_t(i-1) + P_t(i+1) - 2P_t(i).$$

Similar arguments also apply to the case of several particles. Thus,

$$\begin{aligned} \frac{d}{dt} E \prod_{i=1}^k \xi_0(\eta_t^i(x_i)) &= \sum_{i_1 < \dots < i_k} Pr(\xi_0(i_1) = 1, \dots, \xi_0(i_k) = 1) \frac{d}{dt} P_t(i_1, \dots, i_k) \\ &= \sum_{i_1 < \dots < i_k} Pr(\xi_0(i_1) = 1, \dots, \xi_0(i_k) = 1) \left\{ \sum_{j=1}^k \Delta_{i_j}^2 P_t(i_1, \dots, i_k) \right\} \\ &= \sum_{i_1 < \dots < i_k} \left\{ \sum_{j=1}^k \Delta_{i_j}^2 Pr(\xi_0(i_1) = 1, \dots, \xi_0(i_k) = 1) \right\} P_t(i_1, \dots, i_k) \\ &= \sum_{j=1}^k \Delta_{x_j}^2 E \prod_{i=1}^k \xi_0(\eta_t^i(x_i)). \end{aligned}$$

Clearly, boundary conditions are the same as for the original exclusion process:

$$E\xi_0(\eta_t^1(0)) \prod_{i=1}^{k-1} \xi_0(\eta_t^i(x_i)) \equiv 0,$$

$$E \prod_{i=1}^{k-1} \xi_0(\eta_t^i(x_i)) \xi_0(\eta_t^i(S+1)) = E \prod_{i=1}^{k-1} \xi_0(\eta_t^i(x_i)).$$

Therefore, the process that we have constructed is indeed dual to the original exclusion process. Summing up and taking into account the fact that all particles finally get into one of the two absorbing states, we obtain

$$\begin{aligned} m_k(\infty; \{x_1, \dots, x_k\}) &= E \prod_{i=1}^k \xi_\infty(x_i) = E \prod_{i=1}^k \xi_0(\eta_\infty^i(x_i)) \\ &= Pr(\eta_\infty^1(x_1) = S+1, \dots, \eta_\infty^k(x_k) = S+1) = P_\infty^{(k)}, \end{aligned}$$

where  $P_\infty^{(k)}$  is the probability that all particles in the dual process reach the absorbing state  $S+1$ .

### 3.3 Analysis of the Dual Process

Consider the exclusion process  $\eta_t = \eta_t^{(\infty)} = (x_t, y_t)$  for two particles on the segment  $[0, S+1]$ . At the initial time instant, the particles are at points  $0 < x_0(S) < y_0(S) < S+1$ , respectively, such that  $\frac{x_0(S)}{S+1} \rightarrow_{S \rightarrow \infty} \alpha$  and  $\frac{y_0(S)}{S+1} \rightarrow_{S \rightarrow \infty} \beta$ . It is also convenient to assume that the distance between them is at least 2. Clearly, this does not lose generality. Let us show that

$$P_\infty^{(2)} \rightarrow_{S \rightarrow \infty} \alpha\beta,$$

whence the theorem follows.

For an exclusion process  $\eta_t$ , we introduce time intervals  $(T_k, T'_k)$  when particles are at distance 1 from each other, assuming that the rest of the time the distances are greater than 1. We call  $T_k$  the  $k$ th meeting moment. Using these time moments, for  $k = 0, 1, \dots$  we define processes  $\eta^{(k)}(t)$  (which are non-Markovian if  $k > 0$ !). In each of the processes, two particles with the same initial disposition are walking.

In the process  $\eta^{(0)}(t)$ , particles walk independently (simple symmetric random walk) and do not see each other. In the process  $\eta^{(k)}(t)$ , the exclusion process works up to time moment  $T'_k$ . After that the particles become at distance 2 from each other, and from this point they start walking independently and do not see each other.

Let  $P_k$  be the probability that in the process  $\eta^{(k)}(t)$  both particles come to the point  $S+1$  before one of them hits the origin. Since the number of particle meetings before absorption is finite with probability 1, it suffices to prove that for large  $S$  the probabilities  $P_k$  differ little from  $P_0$ .

First of all, it is easy to prove that asymptotically as  $S \rightarrow \infty$  we have

$$P_0 \rightarrow \alpha\beta.$$

Our main method is recurrence relation between the probabilities  $P_k$ .

First we compare  $P_1$  and  $P_0$ . Define a random variable  $\tau(x, y)$ , the moment of the first meeting of the particles provided that they started from the points  $x$  and  $y$ . It is clear that for all processes  $\eta^{(k)}(t)$ ,  $k > 0$ , the distribution of this variable is the same. Let  $x_\tau$  and  $y_\tau = x_\tau + 1$  be points where the first and second particles, respectively, are at the first meeting moment, and let  $\mathbb{P}_0(x_\tau = n)$  be the probability that the first particle is at point  $n$  at this meeting moment. Let  $P_k(x, y)$  be the probability for two particles in the  $k$ th problem to reach  $S + 1$  if at the initial time moment they are at points  $x$  and  $y$ . Note also that the event  $\bigcup_{n=1}^S (x_\tau = n)$  comprises also the outcome that the two particles reach  $S + 1$  earlier than die, since in this case the distance between them would necessarily be equal to one, at least at the points  $(S, S + 1)$ .

Then, by the total probability formula,

$$P_1 = \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_\tau = n) (P_0(n, n+2) + P_0(n-1, n+1)) + \frac{S}{S+1} P_0(x_\tau = S).$$

We transform the sum in parentheses, first rewriting each term:

$$\begin{aligned} P_0(n, n+2) - P_0(n, n+1) &= \frac{n(n+2)}{(S+1)^2} - \frac{n(n+1)}{(S+1)^2} = \frac{n}{(S+1)^2}, \\ P_0(n-1, n+1) - P_0(n, n+1) &= \frac{(n-1)(n+1)}{(S+1)^2} - \frac{n(n+1)}{(S+1)^2} = -\frac{n+1}{(S+1)^2}, \end{aligned}$$

and therefore

$$P_0(n, n+2) + P_0(n-1, n+1) = 2P_0(n, n+1) - \frac{1}{(S+1)^2}.$$

Thus, we finally obtain an expression for  $P_1$  via  $P_0$ :

$$\begin{aligned} P_1 &= \sum_{n=1}^{S-1} P_0(x_\tau = n) \left( P_0(n, n+1) - \frac{1}{2(S+1)^2} \right) + \frac{S}{S+1} P_0(x_\tau = S) \\ &= \sum_{n=1}^S P_0(x_\tau = n) P_0(n, n+1) - \frac{1}{2(S+1)^2} \sum_{n=1}^{S-1} P_0(x_\tau = n) \\ &= P_0 - \frac{C_1}{2(S+1)^2}. \end{aligned}$$

To get the last equality, we again use the total probability formula. Note that  $C_1 = \sum_{n=1}^S P_0(x_\tau = n)$  is a positive constant less than one, since this sum is the



probability that the particles meet at least once before one of them is absorbed at 0 or  $S + 1$ .

Now, taking the obtained result into account, we make similar reasoning for the probability  $P_2$ :

$$\begin{aligned} P_2 &= \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_\tau = n) (P_1(n, n+2) + P_1(n-1, n+1)) + \frac{S}{S+1} P_0(x_\tau = S) \\ &= \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_\tau = n) \left( P_0(n, n+2) + P_0(n-1, n+1) \right. \\ &\quad \left. - \frac{1}{2(S+1)^2} (C_1(n, n+2) + C_1(n-1, n+1)) \right) + \frac{S}{S+1} P_0(x_\tau = S), \end{aligned}$$

where

$$C_1(i, j) = \sum_{n=1}^{S-1} P_0(x_{\tau(i,j)} = n).$$

Denote

$$C_2 = \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_\tau = n) (C_1(n, n+2) + C_1(n-1, n+1));$$

then

$$P_2 = P_1 - \frac{C_2}{2(S+1)^2}.$$

Taking into account that  $C_1 < 1$ , we roughly estimate  $C_2$ :

$$C_2 = \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_\tau = n) (C_1(n, n+2) + C_1(n-1, n+1)) < \sum_{n=1}^{S-1} \mathbb{P}_0(x_\tau = n) < 1.$$

Analogously, a similar relation can be written for any  $k$ :

$$P_k = P_{k-1} - \frac{C_k}{2(S+1)^2},$$

where

$$C_k(i, j) = \frac{1}{2} \sum_{n=1}^{S-1} P_0(x_{\tau(i,j)} = n) (C_{k-1}(n, n+2) + C_{k-1}(n-1, n+1)),$$

$$C_k := C_k(\alpha, \beta), \quad C_k < 1.$$

We rewrite this as follows:

$$P_k = P_0 - \frac{1}{2(S+1)^2} \sum_{i=1}^k C_i.$$

Passing to the limit as  $k \rightarrow \infty$ , we obtain

$$P_\infty = P_0 - \frac{1}{2(S+1)^2} \sum_{i=1}^{\infty} C_i.$$

Thus, we have an expression for the difference of probabilities

$$P_0 - P_\infty = \frac{1}{2(S+1)^2} \sum_{i=1}^{\infty} C_i.$$

It remains to estimate this difference and show that it tends to zero as  $S \rightarrow \infty$ .

By the construction,  $C_k$  is the probability that two independently walking particles meet at least  $k$  times before one of them is absorbed. To estimate this quantity, we introduce an auxiliary model.

Consider a process  $X_t \in [0, S]$ , symmetric random walk of a particle on the segment  $[0, S]$  with  $X_0 = 1$ . Let  $\gamma_k$  be the probability for a particle to come to zero  $k$  times before reaching  $S$ . We claim that  $C_k$  is not greater than  $\gamma_k$ . Indeed,  $X_t + 1$  can be interpreted as the distance between two particles in our original problem at time instant  $t$ . The condition  $X_0 = 1$  means that at the initial time moment we place the particles at the smallest distance before their meeting, i.e., at distance 2. If  $X_t = S$ , one of the particles is absorbed for sure, since the distance is equal to  $S + 1$ . Therefore, if two independently walking particles meet  $k$  times before one of them is absorbed, then the auxiliary particle  $X_t$  hits zero  $k$  times before it comes to  $S$ . Thus, the probability to hit zero  $k$  times in the auxiliary model is (at least) not less than the probability to meet  $k$  times in the original problem.

Let us calculate  $\gamma_k$ . It is clear that  $\gamma_1 = \frac{S-1}{S}$ . If a particle comes to 0, it makes the next step to the right, to the point 1, and everything starts over again; hence,

$$\gamma_k = \gamma_{k-1} \frac{S-1}{S}.$$

Finally, we obtain

$$\gamma_k = \left( \frac{S-1}{S} \right)^k.$$

Now we can again estimate the difference between  $P_0$  and  $P_\infty$ :

$$\begin{aligned} P_0 - P_\infty &= \frac{1}{2(S+1)^2} \sum_{i=1}^{\infty} C_i \leq \frac{1}{2(S+1)^2} \sum_{i=1}^{\infty} \gamma_i \\ &= \frac{1}{2(S+1)^2} \sum_{i=1}^{\infty} \left( \frac{S-1}{S} \right)^i = \frac{S-1}{2(S+1)^2} = \frac{1}{2(S+1)} - \frac{1}{(S+1)^2}. \end{aligned}$$

Therefore,  $[b]|P_\infty - P_0| = O\left(\frac{1}{S+1}\right)$  as  $S \rightarrow \infty$ ; since  $[b]P_k \rightarrow_{k \rightarrow \infty} P_\infty$  for any  $S$ , this completes the proof.

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